# Sustainability in General Equilibrium

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**Abstract**

Climate change, resource depletion, nuclear energy policy, and the solvency of national pensions have raised questions about the sustainability of current decisions. One definition of sustainability is ensuring that the welfare of future generations is not expected to decrease. Viewed this way, sustainability can be thought of as a constraint on economic decision-making. We examine the impact of such a constraint in a general equilibrium economy with a risky asset and show that the implications can differ substantially from those derived in a partial equilibrium setting. In general equilibrium, sustainability boils down to supply-side factors such as economic growth and consumption risk, with increased growth and decreased risk allowing for a higher consumption-wealth ratio. In partial equilibrium, sustainability is determined by the risk-free rate and an adjustment for risk, with risk increasing the sustainable consumption-wealth ratio. Here, the positive drift in expected wealth due to the risk premium more than offsets the negative impact of risk on expected welfare. In general equilibrium, however, the risk-free rate adjusts, and risk makes it harder to achieve the sustainability criterion. We explore the implications of this result for sustainability policies, with a focus on endogenous investment, catastrophic risk management, financial frictions, and the supply of risk-sharing assets, which we define as "financial depth."

**Keywords:** growth, rare disasters, climate risk, long-run risk, welfare, intergenerational equity, financial depth.

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# 1 Introduction

In October of 2022, members of the Last Generation group threw mashed potatoes at one of Monet's famous haystack paintings in a museum in Potsdam, Germany. The action was meant to highlight the urgency of addressing climate change, despite the distractions of more immediate challenges such as energy scarcity and global inflation. More generally, the protests reflect a tension between the welfare of current generations, who may be more focused on the needs of the present, and the welfare of future generations, who will bear the consequences of delayed action on everything from carbon emissions to infrastructure to national debt.

This tension raises fundamental questions about the generational ethics of household, firm, and government decisions about spending and investment. What determines whether current decisions are in some sense "sustainable"? The standard approach in economics has focused less on the sustainability of current consumption and more on the question of whether current decisions maximize the discounted expected present value of current and future utility. As is well known, the optimality of decision-making in this setting depends crucially on the choice of discount rate, with controversy surrounding the ethics of discounting the welfare of future generations at all.[1]

An alternative approach defines decisions as "sustainable" if they provide for a non-decreasing expected path of future welfare. This notion of sustainability reflects the language of the report by the World Commission on Environment and Development (1987), which defined sustainable development as "[meeting] the needs of the present without compromising the ability of future generations to meet their own needs." In this sense, the criterion of sustainability operates as an additional constraint on the choices facing society. Arrow et al. (2004) apply this criterion to various regions around the world and find mixed evidence for the sustainability of consumption and investment decision, with poorer regions showing declining investment in a broad measure of the productive base ("genuine wealth") and rich countries showing positive growth.

The criterion in Arrow et al. (2004) does not account for capital market risk. Campbell and Martin (2022) introduce a source of risky capital and show that the sustainable social rate of time preference (equal to the consumption-wealth ratio) lies between the risk-free rate and the risky return on capital. In their model, risky capital provides additional space for the sustainable consumption-wealth ratio, allowing it to rise above the risk-free rate. In a setting with exogenous

[1] See Dasgupta (2008) and Dasgupta (2021) for a review of the literature on discounting. Arguments favoring the use of a low discount rate for the distant future include Dybvig et al. (1996), Weitzman (1998), and Gollier (2002). In the context of climate change, the Stern Report (Stern, 2007) has argued for the use of effectively a zero discount rate, which has received criticism from Nordhaus (2007), among others.

rates of return, with a safe but low-returning investment, and a risky but high-returning investment, the economy allocates capital correctly (the sustainability constraint does not distort the investment decision), and the presence of risky investments makes it easier for an economy to meet its sustainability threshold.

That framework, however, operates in a partial-equilibrium world. Given the broad interpretation of their sustainability framework, which extends to both the national level and the world as a whole, it's natural to wonder how things would change in general equilibrium, where asset prices, consumption decisions, and economic growth are all interconnected. This question motivates the current paper. We introduce a sustainability constraint into a general equilibrium framework and show that it matters for the fundamental question of whether a current set of consumption and portfolio decisions is sustainable.

Considering general equilibrium provides new insights into the determinants of sustainability. In partial equilibrium, the maximum sustainable level of the consumption-to-wealth ratio exceeds the risk-free rate by a risk term equal to half the squared Sharpe ratio. The risk-free rate is exogenous in partial equilibrium, and the risk premium ensures that wealth has positive expected drift. This positive drift term more than offsets the negative impact of risk on expected welfare due to risk aversion, so that the presence of risk has a net positive effect on the sustainability of consumption.

In general equilibrium, however, the risk-free rate adjusts, and sustainability is determined entirely by supply-side factors, such as economic growth and economic volatility. Growth is positive for sustainability, while risk of all kinds is negative; sustainability requires that economic growth be high enough to offset the impact of risk. In the case of Brownian volatility, we show that the risk term has only a second-order impact on sustainability, but that rare disaster risk has the potential to overwhelm the combined impact of growth and higher-frequency risk. We also study the impact of long-run Markov risk and show that in both settings—in partial and in general equilibrium—non-stationary risk makes the sustainability constraint more likely to bind.[2] Though simple, the general equilibrium condition highlights the fundamental role of supply-side factors in determining the sustainability of an economy. It also points to a different set of policy implications than emerge in the partial equilibrium setting.

We explore some of these policy implications by allowing for endogenous growth, catastrophic

[2]That long-run risk worsens prospects for sustainability even in partial equilibrium is a major distinction from Campbell and Martin (2022), who find that Brownian and disaster risk—via risk premia—provides additional space for sustainability. Our results highlight that the type of risk matters critically for sustainability implications.

risk mitigation, financial market depth, and financial market integration. Growth plays a fundamental role in our general equilibrium sustainability condition. In order to examine the impact of growth policies, we endogenize growth through a simple AK model that allows investment in physical or human capital to raise growth rates. The message here is straightforward: any policies that improve growth outcomes will push an economy further above its sustainability threshold.

On the risk side of the equation, we examine policies that affect both the frequency and impact of rare disasters (in the spirit of Barro, 2006), as well as the role of higher-frequency risks captured by our Brownian process. Given the large potential impact of disaster risk on sustainability, we discuss policies that influence both the arrival rate of disasters, as well their magnitude. Canonical examples of such policies include climate and nuclear treaties, as well as mitigation strategies. Managing higher-frequency risks, however, points toward financial solutions, including reducing financial frictions, improving financial depth, or optimizing economic openness. Following Caballero, Farhi, and Gourinchas (2008), for instance, we model financial depth as a fraction of future cashflows that can be capitalized as assets. We find that the higher this fraction, the easier it is for countries to meet the sustainability constraint.

The impact of financial integration, however, depends on the financial depth of the respective countries. In this case, market risk pushes down the equilibrium risk-free rate, making it harder to achieve the sustainability criterion. Further, because countries differ in their economic growth experience and financial markets, they also have fundamentally different thresholds for sustainable consumption. Even if the world agreed on the science of climate change, the economics of climate change are inescapably country-specific. A country with high growth may be less concerned about the impact of current decisions, even if they result in increased carbon emissions. And this would be true even if they bore the full brunt of pollution.

The emphasis on generational equity has been most sharply articulated in the context of climate change. And indeed, the role of catastrophic risk in the model underscores the urgency of reducing the frequency and impact of climate events. But the general equilibrium framework also highlights the importance of macroeconomic policies in ensuring that future generations, at least in expectation, enjoy at least as high a welfare as those present. It is increasingly unlikely that the world will meet the Paris Agreement's target of limiting the mean global temperature increase in this century to 2° Celsius, even with aggressive policies aimed at reducing carbon emissions. In addition to addressing climate change, the general equilibrium sustainability constraint points toward policies that might still create the conditions for rising welfare in the future: namely, higher economic growth, improved risk-sharing, and greater openness.

In studying the general equilibrium implications of a sustainability constraint, we are building on a literature that goes all the way back to Ramsey (1928) and the question of how much an economy should save for the future. The intergenerational justice implications of such saving decisions were developed in (among others) Rawls (1999), Arrow (1973), Solow (1974), and Hartwick (1977). Dasgupta (2021) provides a comprehensive summary of the philosophical and economic tensions involved in any question of intertemporal welfare. Not surprisingly, some of the thorniest issues center on the question of the appropriate discount rate, with Stern (2007) applying a near-zero discount rate to argue for drastic changes in the present, and others highlighting the implications and hazards of low discount rates (Nordhaus, 2007), the need to balance the costs and benefits of any sustainability strategy (Nordhaus, 1991), and the importance of modeling catastrophic risk (Weitzman, 2007).

More directly, there are a number of studies that formalize the ethical requirement that either consumption or welfare should not decrease over time. We have already mentioned Campbell and Martin (2022) and Arrow et al. (2004), which are the closest to our approach. These studies, however, build on a series of contributions that address different aspects of sustainable welfare conditions, including Pezzey (1992), Howarth (1995), Solow (1995), Dietz and Asheim (2012), and Campbell and Sigalov (2022). As is often the case, Solow (1995) provides an especially eloquent case for a sustainability constraint: "The duty imposed by sustainability is to bequeath to posterity not any particular thing [...] but rather to endow them with whatever it takes to achieve a standard of living at least as good as our own and to look after their next generation similarly." Our contribution is to apply this standard in a general equilibrium setting with different sources of risk and policy levers.

The rest of the paper proceeds as follows. Section 2 compares the sustainability condition in a partial-equilibrium setting and a general-equilibrium one, with Brownian volatility. Section 3 then shows how these results depend on the presence of disaster risk and a persistent Markov regime-switching process. Section 4 presents extensions that endogenize the most important factors in the general-equilibrium sustainability condition: growth and financial development. Section 5 discusses various policy options aimed at relaxing the sustainability constraint, and the final section concludes.

## 2 Sustainability in partial and general equilibrium

The notion of "sustainability" has taken on a variety of meanings in the popular press, policy circles, and academic journal articles. What does it mean to "meet the needs of the present without compromising the ability of future generations to meet their own needs" (World Commission on Environment and Development, 1987)? At the broadest level, it means that the current path of decisions leaves future generations at least as well off as the current generation. In this interpretation, sustainability is less about optimizing current decisions than about satisfying a constraint on the trajectory of expected welfare across future generations. This is the approach taken by Arrow et al. (2004) and Campbell and Martin (2022), and it is the starting point for our own analysis.

### 2.1 Partial-equilibrium sustainability constraint

As in Campbell and Martin (2022), we begin by considering a standard continuous-time optimization model, in which a representative investor chooses consumption and a risky portfolio share to maximize the expected discounted value of lifetime utility. Investment risk takes the form of a Brownian process, an assumption that we will relax in Section 3. We begin with the simplest case of log utility and then extend the results to CRRA and recursive preferences.

A representative investor discounts the future at rate $\rho$ and chooses consumption $c_t$ and the risky portfolio share $\alpha_t$ to maximize expected discounted lifetime utility:

$$\mathbb{E} \int_0^\infty e^{-\rho t} \log c_t \, \mathrm{d}t.$$

There is a risk-free asset with return $r$ and a risky asset with Brownian volatility $\sigma$ and excess return $\mu$. Wealth $W_t$ therefore follows the process

$$\frac{\mathrm{d}W_t}{W_t} = (r + \alpha_t \mu - \theta_t) \, \mathrm{d}t + \alpha_t \sigma \, \mathrm{d}Z_t,$$

where $\theta_t \equiv c_t / W_t$ is the consumption-wealth ratio and $Z_t$ is a Wiener process. Letting $V(W_t)$ denote the value function, the Hamilton-Jacobi-Bellman (HJB) equation is:

$$\rho V = \max_{c_t, \alpha_t} \left\{ \log(c_t) + V'(r + \alpha_t \mu - \theta_t) W_t + \alpha_t^2 \frac{\sigma^2}{2} W_t^2 V'' \right\}. \tag{1}$$

With log utility, the value function can be written as

$$V(W_t) = A \log W_t + b,$$

where $A$ and $b$ are constants, so that $V(W_t)' = A/W_t$ and $V(W_t)'' = -A/W_t^2$.

As is standard, the optimal consumption is $c = \rho W$, and the optimal share in the risky asset is $\alpha = \mu/\sigma^2$. Substituting these optimal values into equation 1 above, and using the general form for the value function with log utility, we can solve for welfare as:

$$V_t = \frac{\log W_t}{\rho} + \frac{\log(\rho)}{\rho} + \frac{1}{\rho^2}\left(r - \rho + \frac{SR^2}{2}\right), \tag{2}$$

where $SR \equiv \mu/\sigma$ denotes the Sharpe ratio.

Since welfare only depends on time through wealth, the expected change in welfare is just the expected drift in log wealth divided by the discount rate, $\rho$. Letting $\nu \equiv \mathbb{E}[\mathrm{d}\log W_t]$ denote the expected drift in log wealth, we have

$$\mathbb{E}[\mathrm{d}V_t] = \frac{\mathbb{E}[\mathrm{d}\log W_t]}{\rho} \equiv \frac{\nu}{\rho}.$$

The sustainability constraint from Campbell and Martin (2022) requires that welfare cannot have a negative drift, which amounts to $\nu \geq 0$. By Itô's Lemma, the evolution of log wealth is

$$\begin{aligned}
\mathrm{d}\log W_t &= \frac{\mathrm{d}W_t}{W_t} - \frac{1}{2}\left(\frac{\mathrm{d}W_t}{W_t}\right)^2, \\
&= \left(r - \theta + \alpha\mu - \alpha^2\sigma^2/2\right)\mathrm{d}t + \alpha\sigma \mathrm{d}Z,
\end{aligned}$$

since $(\mathrm{d}W_t/W_t)^2 = \alpha^2\sigma^2\mathrm{d}t$. Hence, the expected drift $\nu$ is given by

$$\nu = r - \theta + \underbrace{\alpha\mu}_{\substack{\text{return}\\ \text{adjustment}}} - \underbrace{\alpha^2\sigma^2/2}_{\substack{\text{utility}\\ \text{adjustment}}}. \tag{3}$$

It is worth noting how risk affects the expected change in welfare in partial equilibrium. First, risk operates through a return adjustment relative to the risk-free rate. Risky investments earn a risk premium, which increases the drift in wealth and generates a higher average return. All else equal, a higher risk premium is good for sustainability because it increases the expected trajectory of resources available in the future.

Second, risk operates through a utility adjustment. Relative to the drift in wealth, which is $r + \alpha\mu - \theta$, the drift in welfare is lower by the term $\alpha^2\sigma^2/2$. This utility adjustment is due to the curvature in the welfare function (risk aversion). With concave utility, higher risk decreases the expected value of future welfare (Jensen's inequality). This force puts downward pressure on the expected trajectory of welfare. Whether risk increases or decreases the drift of welfare depends on whether the return adjustment outweighs the utility adjustment.

At the optimal portfolio share $\alpha = \mu/\sigma^2$, the risk premium term equals $SR^2$, while the welfare adjustment is $-SR^2/2$. Evaluating the expected drift at the optimal values of consumption and the risky share, we have

$$\nu = r - \rho + \frac{1}{2}SR^2.$$

In partial equilibrium, the net impact of risk on the expected welfare trajectory is positive: the risk premium more than offsets the negative impact of risk on welfare. This is one of the main lessons from Campbell and Martin (2022): namely that risk allows for a positive drift in wealth, even with a binding sustainability constraint. From the equation above, sustainability in partial equilibrium requires

$$r + \frac{1}{2}SR^2 \geq \rho. \tag{4}$$

Despite the relatively simple setting (log utility and Brownian volatility), we can already see an important dynamic with the sustainability constraint in partial equilibrium. The drift in welfare increases with the risk-free rate, patience, and the Sharpe ratio. The constraint is likely to bind whenever the discount rate is high relative to the risk-free return and when the Sharpe ratio is low. Noting that consumption equals the discount rate times wealth, we can solve for the constrained consumption-wealth ratio, $\theta_{\text{con}}$ as:

$$\theta_{\text{con}} = r + \frac{1}{2}SR^2. \tag{5}$$

As in Campbell and Martin (2022), the constrained consumption-wealth ratio lies between the risk-free rate and the expected return on optimally invested wealth, $r + SR^2$.

Importantly, this basic intuition about the partial equilibrium result holds for different assumptions about preferences and different assumptions about the nature of risk (see Appendix A for derivations). For example, with Epstein-Zin preferences (Epstein and Zin, 1989; Duffie and

Epstein, 1992) with elasticity of intertemporal substitution (EIS) $\epsilon$ and coefficient of relative risk aversion $\gamma$, the sustainability condition becomes:

$$r - \rho + \frac{SR^2}{2\gamma} \geq 0, \tag{6}$$

which is independent of $\epsilon$. This condition nests the special case of additive CRRA utility, in which case $\epsilon = 1/\gamma$. In either case, the risk term in the sustainability constraint is modified by the level of risk aversion and considering higher risk aversion than log utility ($\gamma > 1$) relaxes the sustainability constraint.

All of these conditions share the common property that the presence of risk provides additional "room" in the sustainability constraint, in the sense that the discount rate can exceed the risk-free rate. This analysis assumes, however, that returns are exogenous.

### 2.2 General-equilibrium sustainability constraint

How do things change in a general equilibrium economy, in which asset prices need to be consistent with consumption and saving decisions? We now endogenize returns in the simplest setting possible, a Lucas (1978) tree economy with a single risky asset and a risk-free asset in zero net-supply. The tree accounts for all the productive factors in the economy. We again assume that the representative consumer has log preferences.

The tree's dividend stream $D_t$ (the "fruit") follows a geometric Brownian motion

$$\frac{dD_t}{D_t} = g\,\mathrm{d}t + \sigma\,\mathrm{d}Z_t, \tag{7}$$

where $g$ is the growth rate, $\sigma$ is the volatility of risk, and $Z_t$ is a Weiner process.

The tree has per-dividend price $Q_t$ (the "asset price"), and the risk-free rate $r_t$ is endogenously determined in equilibrium. In this setting, the asset price $Q_t$ and the risk-free rate $r_t$ need to adjust so that the representative consumer is willing to hold the entire tree and consume the dividends.

Because the tree represents all productive factors in the economy and the risk-free asset is in zero net-supply, aggregate wealth is $W_t = Q_t D_t$. Furthermore, since optimal consumption is $c_t = \rho W_t$, and consumption equals the tree's dividends, $c_t = D_t$, the asset price is stationary:

$$\rho Q = 1 \implies Q = \frac{1}{\rho}.$$

Intuitively, since all wealth is invested in the tree, household wealth grows at the rate of economic growth, $g$.

The consumer's stochastic discount factor, $\Lambda_t \equiv e^{-\rho t}/c_t$, prices all assets (see Cochrane, 2009, 2017). Since consumption equals the tree's dividends, the drift and volatility terms for consumption are the same as those for the dividends. And because consumption is proportional to wealth, the volatility of both consumption and wealth have to equal $\sigma$. In equilibrium the risk premium on the tree equals the volatility of the stochastic discount factor times the volatility of dividends. Hence, the Sharpe ratio of the risky investment equals the volatility of consumption. Since the optimal portfolio allocation sets $\alpha = \mu/\sigma^2$ and $\alpha = 1$ by market clearing, the equilibrium returns on the tree satisfy:

$$SR = \sigma \implies \mu = \sigma^2.$$

By the stochastic maximum principle, the risk-free rate satisfies

$$r_t \, \mathrm{d}t = -E_t \left[ \frac{\mathrm{d}\Lambda_t}{\Lambda_t} \right] = \rho \, \mathrm{d}t + E_t \left[ \frac{\mathrm{d}c_t}{c_t} \right] - Var_t \left[ \frac{\mathrm{d}c_t}{c_t} \right], \tag{8}$$

implying a constant risk-free rate:

$$r = \rho + g - \sigma^2. \tag{9}$$

While this is a standard equation, it is worth understanding the intuition for how risk affects the risk-free rate in general equilibrium. The return on the tree includes the dividend yield from output, which is $1/Q$, plus capital gains due to growth, which is $g$. Since $Q = 1/\rho$ in equilibrium, the total return on investing in the tree is $\rho + g$. The risk premium is $\rho + g - r$. For the tree to earn a risk premium of $\sigma^2$, the risk-free rate must adjust to reflect that risk, so that $\rho + g - r = \sigma^2$, which implies $r = \rho + g - \sigma^2$. Once the risk-free rate adjusts for risk, the return on the household's portfolio is simply $\rho + g$. And so, after consuming at a rate $\rho$, wealth grows at a rate $g$.

This has important implications for sustainability. For a given risk-free rate, the sustainability constraint with log utility requires that $\nu = r - \rho + \frac{1}{2}\sigma^2 \geq 0$. By substituting the risk-free rate in general equilibrium, $r = \rho + g - \sigma^2$, we get our first main result.

**Proposition 1.** *With log utility and Brownian risk, sustainability in general equilibrium requires*

$$g - \frac{1}{2}\sigma^2 \geq 0. \tag{10}$$

Equation (10) is quite different from (4), though both reflect the same constraint on welfare. While the household rate of time preference was prominent in partial equilibrium, it is completely absent in the constraint in general equilibrium. In addition, risk had the property of *relaxing* the constraint in partial equilibrium, while it actually *tightens* the constraint in general equilibrium. Risk $\sigma$ and time-preference $\rho$ are both reflected in the risk-free rate. In general equilibrium, higher risk makes sustainability less likely.

What is the intuition for the contrasting way that risk operates in general equilibrium? Fundamentally, the drift in log wealth can be decomposed into a growth term and a utility adjustment:

$$\nu = \underbrace{\mathbb{E}[\mathrm{d}W_t/W_t]}_{=g} - \underbrace{\sigma^2/2}_{\substack{\text{utility}\\ \text{adjustment}}} . \tag{11}$$

Recall that in partial equilibrium, risk affected the drift in wealth through two channels: a return channel (the risk-premium) and a utility channel (concavity). Fixing the risk-free rate, higher risk led to a higher overall return on the investor's portfolio.

In general equilibrium, the return on household wealth is determined entirely by fundamentals: wealth grows at the rate of economic growth, which is equivalent to the expected growth rate of the tree, $g$. The risk-free rate adjusts to absorb the risk premium. Thus, the return channel is entirely absent in general equilibrium; higher risk does not lead to higher returns overall since expected returns are determined entirely by the underlying growth rate of the economy. Because the return channel is completely absent in general equilibrium, risk operates only through the utility adjustment due to concavity. Thus, in general equilibrium, risk makes the sustainability constraint more likely to bind.

As with the partial-equilibrium model, the basic intuition behind the general-equilibrium result does not change with different specifications of preferences (see Appendix A for derivations). For instance, with Epstein-Zin preferences, the risk-free rate satisfies

$$r = \rho + \frac{1}{\epsilon} g - \frac{\gamma\left(1+\frac{1}{\epsilon}\right)}{2}\sigma^2,$$

where $g$ and $\sigma$ are the expected growth rate and volatility of consumption, and $\epsilon$ and $\gamma$ are again the EIS and coefficient of risk aversion, respectively. The excess risk-premium equals $\gamma\sigma^2$ which means $SR = \gamma\sigma$. Substituting these into the sustainability constraint, we have:

$$\epsilon\left(\rho + \frac{1}{\epsilon}g - \frac{\gamma\left(1+\frac{1}{\epsilon}\right)}{2}\sigma^2 - \rho + \frac{\gamma^2\sigma^2}{2\gamma}\right) \geq 0, \tag{12}$$

which reduces to:

$$g - \frac{1}{2}\gamma\sigma^2 \geq 0. \tag{13}$$

Interestingly, the $\epsilon$ drops out and there is only an adjustment for risk aversion, which (intuitively) raises the bar for meeting the sustainability threshold. Note, however, that the opposite occurs in partial equilibrium. In equation (6), higher risk-aversion relaxed the constraint because the returns channel dominated the utility adjustment channel (and higher risk aversion requires a higher risk premium). In contrast, in general equilibrium higher risk-aversion tightens the sustainability constraint because higher risk aversion magnifies the utility adjustment channel.

The sustainability requirement in general equilibrium is markedly different from the one in partial equilibrium, and it yields several key insights. First, sustainability in general equilibrium depends only on a combination of economic growth and economic volatility ($g$ and $\sigma$). It does not at all depend on the discount rate, $\rho$. To the extent that the general equilibrium framework captures the fundamental forces facing the economy, this has important implications for any policies aimed at avoiding a binding sustainability constraint. Policies that favor economic growth and/or result in lower volatility will make it more likely that an economy is sustainable. This places the emphasis on factors that have been central in the literatures on endogenous growth, macro-finance, and environmental sustainability.

Second, for plausible values of growth and Brownian volatility, the sustainability requirement is unlikely to bind for most economies. Taking the standard deviation of GDP growth as an empirical analogue of $\sigma$, most countries have $\sigma$ values well below 10%, making the risk term second-order relative to GDP growth itself. For example, developed countries have GDP growth in the neighborhood of 2% and standard deviations of GDP growth around 2%. The growth term contributes 2% to the sustainability condition, while the volatility term is 0.02%, or two orders of magnitude smaller in absolute value. Even if we assume a relatively high value of risk aversion, which makes it harder to meet the sustainability threshold, the growth term would still swamp the risk-adjusted volatility measure. As we will see, risk can indeed play an important role if we introduce disaster risk to the model, but in the case of pure Brownian volatility, risk plays only a minor role in the general equilibrium sustainability condition.

Finally, economies can still achieve sustainability through lower consumption. In particular, if we set consumption to $\theta_{\text{con}} = \rho + g - \frac{1}{2}\sigma^2 < \rho$, then the risk-free rate is unaffected because consumption growth and volatility are the same and the rate of preference is the same.[3] Viewed broadly, this suggests a role for policies aimed at increasing national saving.

## 3 Disaster risk and long-run risk

The results so far have assumed that risk takes the form of Brownian volatility. This may understate, however, the true nature of risk facing the world. Some of the greatest threats facing humanity—war, disease, environmental catastrophes—occur infrequently but with a severity and duration that may not be well captured by our Brownian process. In this section, we introduce two alternative specifications of risk: first a jump process similar to the one in Campbell and Martin (2022) ("disaster risk"), and then a Markov switching process that allows for more permanent shocks to the economy ("long-run risk"). As we will show, both the type and the *stationarity* of risk matters in both partial and general equilibrium.

### 3.1 Disaster risk

Jump risks may capture an important source of volatility, particularly with respect to events related to climate change. Campbell and Martin (2022) consider a flexible specification that includes both sources of risk and find that jump risk does not change the fundamental messages that risk allows the constrained consumption-wealth ratio to exceed the risk-free rate, while preserving the property that the optimal investment decision is unchanged by the sustainability constraint. In general equilibrium, however, we have seen that risk works in the opposite direction and actually makes it more difficult to satisfy the sustainability condition. Furthermore, we have also seen that Brownian volatility is unlikely to be quantitatively important for most countries' sustainability constraints.

Here, we follow the general-equilibrium catastrophic risk model in Pindyck and Wang (2013), who consider the question of how much societies should be willing to pay to avoid this particular type of risk. Catastrophic risk follows a Poisson process with mean arrival rate $\lambda$, with shocks destroying $1 - X$ fraction of capital. To derive closed-form expressions, we suppose that $X$ follows

[3] The inequality follows from the fact that $g < \frac{1}{2}\sigma^2$ when the constraint binds.

a Power distribution in $(0,1)$ with shape parameter $\beta > 0$, so that the density function (pdf) is $\zeta(X) = \beta X^{\beta-1}$, implying $\mathbb{E}[X] = \beta/(1+\beta)$.

Following the standard methodology, we can write the real interest rate $r$ and the risk premium $rp$ as

$$r = \rho + \gamma g - \frac{\gamma(1+\gamma)}{2}\sigma^2 - \lambda \mathbb{E}\left[(X^{-\gamma} - 1)\right],$$

and

$$rp = \gamma\sigma^2 + \lambda \mathbb{E}\left[(1-X)(X^{-\gamma} - 1)\right].$$

As shown in Appendix B, the investor's consumption-wealth ratio $\theta$ is

$$\theta = \rho + (\gamma - 1)\left(g - \frac{\gamma}{2}\sigma^2\right) - \lambda\left(\mathbb{E}\left[X^{1-\gamma} - 1\right]\right), \tag{14}$$

and the sustainability constraint with $\alpha = 1$ is

$$r - \theta + \hat{\mu} - \frac{1}{2}\gamma\sigma^2 + \frac{\lambda}{1-\gamma}\mathbb{E}\left[X^{1-\gamma} - 1\right] \geq 0, \tag{15}$$

where $\hat{\mu} = rp + \lambda \mathbb{E}\left[(1-X)\right]$. We can derive the general-equilibrium version of this sustainability constraint by substituting the values of $\theta$, $r$, $\hat{\mu}$, and $rp$ from above into (15).

**Proposition 2.** *With CRRA utility with risk-aversion $\gamma$ and disaster risk, sustainability in general equilibrium requires*

$$g - \frac{1}{2}\gamma\sigma^2 - \lambda\frac{2\gamma - 1}{\beta - (\gamma - 1)} \geq 0. \tag{16}$$

As in Proposition 1, risk decreases the sustainability constraint in equilibrium, and for the same reason. While disaster risk increases the risk premium on the risky asset, it also decreases the risk-free rate. There is no returns channel (i.e., the positive effect due to the risk premium) because the expected return on wealth is still the expected growth rate of output. Meanwhile, the utility adjustment due to concavity is strictly negative and exacerbated by (larger) disaster risk. Thus, in general equilibrium, increasing risk—whether by increasing $\sigma$ or by explicitly accounting for disaster risk—worsens sustainability.

While disaster and Brownian risk operate conceptually in the same way, explicitly incorporating disaster risk can have quantitative implications. We have already seen that the Brownian risk term has only a second-order impact on the sustainability constraint. The impact of disaster risk, however, depends on the value of risk aversion, the frequency of Poisson shocks and the shape parameter governing the size of the shocks. As a starting point, we can take the calibration in Pindyck and Wang (2013), which sets $\gamma = 3$, $\lambda = 0.734$, and $\beta = 23$. With these parameters, the disaster risk term, $\lambda(2\gamma - 1)/(\beta - (\gamma - 1))$, is 17.5%, which will swamp the impact of growth and Brownian volatility in the sustainability constraint. This calibration, however, is geared toward equity risk and yields relatively frequent disasters, occurring every 1.4 years ($1/\lambda$).

If we instead consider disaster risks stemming from climate change, it might be more reasonable to imagine events that happen every 50 years, which corresponds to a $\lambda = 0.02$. Maintaining the assumption that $\beta = 23$, which yields approximately a 9% probability of losing at least 10% of output, we obtain a disaster risk term of about 0.5%, which would not be enough to cause the sustainability constraint to bind on an economy with a 2% growth rate and 2% volatility.

The sensitivity of these results to changes in the parameterization of disaster risks suggests that there may be considerable variation in the sustainability of different economies. For example, given moderate amounts of risk, a country with slower growth, such as the U.S., could be in danger of reaching its sustainability constraint, while a faster-growing country, such as China, might be safely above its sustainability constraint. These different experiences may help explain why some countries may be more willing than other to commit to international climate agreements. The slower-growth countries may recognize that they are at risk of leaving future generations with lower welfare, while higher-growth countries are almost certain to leave future generations with higher welfare.

The lesson here is that the calibration of disaster risk plays a crucial role in determining the sustainability of an economy. More frequent and more severe catastrophic risks make it less likely that a given economy is on a sustainable path. It also suggests that there could be a substantial return to mitigating or avoiding severe downside risks facing the climate and economy. One type of risk that is not captured in the models so far, however, is the path dependence of adverse outcomes. Climate models often have "tipping points," beyond which climates and economies suffer irreversible changes. It is worth considering how these kinds of risks affect the sustainability criteria in both partial and general equilibrium.

### 3.2 Long-run risk

Disaster risk admits the possibility of large, discontinuous shocks within a stationary framework. These shocks can be severe, but they are also transitory. Scientists, however, have documented a growing list of potentially irreversible "tipping points" that could both amplify current threats to climate change and have lasting effects on the planet (Lenton et al., 2008). A formal model of climate tipping points is beyond the scope of this paper, but we can examine how non-stationary, long-run risks affect the likelihood that a given consumption path is sustainable.

Pritchett (2003) and Jerzmanowski (2006) have stressed that economic performance is not just a story about average growth rates over time, but about different growth regimes that influence growth rates over long periods of time. The possibility of regime change introduces the possibility of poverty traps, stagnation, and prolonged expansions. In the context of sustainability, similar dynamics may apply to tipping points in the environment or breakthroughs in technology that open up lasting opportunities for consistent growth. In order to examine the impact of these sustained risks, we introduce a Markov switching model in the spirit of Chari et al. (1996). As in their model, we can think about an economy that switches between a "good regime," in which distortions are low, and a "bad regime," in which distortions are severe.

Since the analysis of this kind of risk is new to the literature on sustainability constraints, we first examine the partial equilibrium case and then show how things change in general equilibrium. We find that, in contrast to stationary risk, which relaxes the sustainability constraint in partial equilibrium, long-run risk tightens the sustainability constraint. For tractability we consider only Brownian risk (no disaster risk) and log utility.

#### 3.2.1 Partial equilibrium

The Markov switching process has two states, 0 and 1, which affect the excess return on the risky asset. In particular, we assume that state 1 is the good state, with $\mu_1 > \mu_0$, with regime switching governed by a Poisson process with rate $\lambda$.[4]

We first solve the standard optimization problem. Let the value function in state $i$ be $V_i$. Let $\alpha_i$ denote the portfolio and let $c_i$ denote consumption. The investor HJB is

$$\rho V_i = \max_{c_i,\alpha_i} \left\{ \log(c) + V_i'(r + \alpha\mu_i - c_i/W)W + \alpha_i^2 \frac{\sigma^2}{2} W^2 V_i'' + \lambda \left(V_j - V_i\right) \right\}.$$

Optimizing over $c_i$, and $\alpha_i$, we obtain the familiar conditions: $c_i = \rho W$ and $\alpha_i = \mu_i/\sigma^2$. The presence of Markov shocks does not change the decisions about consumption and portfolio choice.

[4]With some abuse of notation we use $\lambda$ to denote the Poisson arrival rate for either disaster risks or long-run risks.

It does, however, affect welfare, which is given by:

$$V_i = \frac{\log W}{\rho} + \frac{\log(\rho)}{\rho} + \frac{1}{\rho^2}\left(r - \rho + \frac{SR_i^2}{2}\right) - \frac{\lambda}{2\rho^2(\rho+2\lambda)}\left(SR_i^2 - SR_j^2\right). \tag{17}$$

In order to see the impact of the Markov switching, we can define

$$\overline{SR_i^2} \equiv \frac{(\rho+\lambda)SR_i^2 + \lambda S_j^2}{\rho+2\lambda},$$

which is a weighted average of the two Sharpe ratio terms, with higher discount rates putting greater weight on the risk associated with the current Markov state.

$$V_i = \frac{\log W}{\rho} + \frac{\log(\rho)}{\rho} + \frac{1}{\rho^2}\left(r - \rho + \frac{\overline{SR_i^2}}{2}\right), \tag{18}$$

which looks like equation (2), except that the value function depends on the state and the Sharpe ratio is the weighted average of the ratios in the two states.

Before we introduce the sustainability constraint, we characterize welfare for given values of $\theta_i$ and $\alpha_i$. As before, we define $\nu \equiv r + \alpha\mu - \theta - \alpha^2\sigma^2/2$. For a given pair $\{\theta_i, \alpha_i\}$, welfare is given by $V_i = (\log W)/\rho + b_i$, with

$$\rho b_i = \log(\theta_i) + \frac{1}{\rho}\nu_i - \frac{\lambda}{\rho+2\lambda}\left(\log(\theta_i) - \log(\theta_j) + \frac{1}{\rho}\left(\nu_i - \nu_j\right)\right). \tag{19}$$

The expected change in welfare is the drift in wealth, minus the Itô term reflecting risk, plus the Poisson term:

$$\begin{aligned}\mathbb{E}[\mathrm{d}V_i] &= \frac{1}{\rho}\left(r + \alpha_i\mu_i - \theta_i - \frac{1}{2}\alpha_i^2\sigma^2\right) - \lambda(b_i - b_j), \\ &= \frac{\nu_i}{\rho} - \lambda(b_i - b_j),\end{aligned} \tag{20}$$

where

$$(b_i - b_j) = \frac{1}{\rho+2\lambda}\left(\log(\theta_i) - \log(\theta_j) + \frac{1}{\rho}\left(\nu_i - \nu_j\right)\right). \tag{21}$$

Optimality requires $\theta_i = \theta_j = \rho$ and $\nu_i - \nu_j = \frac{1}{2}(SR_i^2 - SR_j^2)$. By equation (21), we have

$$\mathbb{E}[\mathrm{d}V_i] \propto r - \rho + \frac{1}{2}SR_i^2 - \frac{\lambda}{2(\rho+2\lambda)}\left(SR_i^2 - SR_j^2\right).$$

Sustainability requires that the expected drift in welfare is non-negative, which yields:

**Proposition 3.** *In partial equilibrium with long-run risk, the sustainability constraint will bind whenever*

$$r + \frac{1}{2}SR_i^2 - \frac{\lambda}{2(\rho + 2\lambda)}\left(SR_i^2 - SR_j^2\right) < \rho. \tag{22}$$

Whereas Brownian and disaster risk provide more space for sustainability in partial equilibrium, long-run risk pushes in the opposite direction. The possibility of moving to a state with lower returns (lower Sharpe), makes it more likely that the sustainability constraint binds. The Sharpe ratio continues to provide additional space in the sustainability constraint, but the probability of moving to a lower state, captured by $\lambda$, makes sustainability less likely, and this risk is not reflected in the risk premium.

Proposition 3 highlights the importance of understanding precisely what types of risks face the global economy and how they affect sustainability concerns. Stationary risks, even if they concern rare disasters, are reflected in risk-premia and therefore provide additional space for sustainability through the returns channel leading to higher expected wealth growth. In contrast, non-stationary risks such as long-run risks, which may better capture the types of risks inherent in climate change, are not reflected in risk-premia and therefore do not contribute to higher expected returns on wealth. Even in partial equilibrium, long-run risks operate entirely through a welfare adjustment that tightens sustainability considerations.

#### 3.2.2 General equilibrium

The natural way to incorporate Markov switching into the general-equilibrium framework is by supposing that growth rates change with the state, letting $g_1 > g_0$. This has the effect of also allowing $r_i$ to change with the state. Because fundamental risk $\sigma$ determines excess returns, in both states we will have $SR_1 = SR_2 = \sigma^2$. (Note the contrast with partial equilibrium.) With changing growth rates, we have

$$r_i = \rho + g_i - \sigma^2 \implies \nu_i = g_i - \sigma^2/2.$$

We can easily characterize when the economy is on a sustainable path in state 1:

**Proposition 4.** *In general equilibrium with long-run risk, the sustainability constraint will bind in state*

1 *if*

$$g_1 - \frac{1}{2}\sigma^2 - \frac{\lambda}{\rho + 2\lambda}(g_1 - g_0) < 0. \tag{23}$$

Intuitively, the prospect of switching to a low-growth regime makes it more likely that an economy will run up against its sustainability constraint. However, as we discussed in the context of the rare disaster risk, it's plausible that $\lambda$ is not very high—perhaps on the order of 2% per year. In that case, the growth loss $g_1 - g_0$ has to be substantial for the Markov risk alone to create a binding sustainability constraint.

### 3.2.3 Downside Risk

For tractability we have so far assumed that Markov risk was symmetric. In reality, we are likely concerned about the prospect of asymmetric *downside* risk: the possibility of moving permanently into a low-growth regime. Now suppose that with probability $\lambda$ the economy moves from the high-growth state 1 to the low-growth state) 0 and then stays there forever. The results of the previous analysis go through with a minor change to the denominator of the Markov term to reflect that state 0 is an absorbing state.

For expositional purposes, suppose initially that the sustainability constraint does not bind in state 0. As shown in Appendix C, we can solve for the following modified sustainability condition:

**Proposition 5.** *Suppose the sustainability constraint does not bind in* 0*. In general equilibrium, the sustainability constraint will bind in state* 1 *if*

$$g_1 - \frac{1}{2}\sigma^2 - \frac{\lambda}{\rho + \lambda}(g_1 - g_0) < 0. \tag{24}$$

This is nearly identical to the condition in Proposition 4, except that there is a larger coefficient on the growth loss term, $g_1 - g_0$, reflecting the consequences of moving to a permanently lower growth regime.

What is the quantitative significance of long-run downside risk? As a rough calibration, we can set $g_1 = 2\%$, $\sigma = 2\%$, and $\rho = 5\%$. Let $\Delta \equiv g_1 - g_0$ be the growth at risk in switching from $g_1$ to $g_0$. Then for the sustainability constraint to bind we need

$$0.02 - \frac{1}{2}(0.02)^2 < \frac{\lambda}{0.05 + \lambda}(\Delta).$$

Even if we assume a relatively high value of $\lambda = 10\%$, we would need a $\Delta$ of almost 3% in order for the sustainability constraint to bind. Given our assumption of 2% growth in the good state, this would amount to a $-1\%$ growth rate in the bad state. If $\lambda$ is closer to 2%, as we argued before, the difference in growth rates would have to be over twice as large. Regardless of the precise calibration, however, it is clear that long-run risk makes it more likely that the sustainability constraint binds.

The analysis so far has assumed that the sustainability constraint does not bind in state 0. But for reasonable calibrations of the model's parameters, we have seen that $g_0 \leq 0$, in which case the sustainability constraint *will always bind* in state 0. The optimal consumption in state 0 is then adjusted to

$$\theta_0 = \rho + g_0 - \frac{\sigma^2}{2} < \rho,$$

which means that $\log(\theta_0)$ is lower than $\log(\rho)$, implying an additional welfare loss when the Markov shock occurs. Importantly, the failure to meet the sustainability constraint after downside risk realizes increases the urgency to pursue sustainability in the present. This dynamic makes the sustainability constraint in state 1 even more likely to bind. It immediately follows that if the sustainability constraint does bind in state 0, we have to update the sustainability constraint in state 1 as follows.

**Proposition 6.** *Suppose $g_0 \leq \sigma^2/2$ so that the sustainability constraint binds in state 0. In general equilibrium, the sustainability constraint will bind in state 1 if*

$$g_1 - \frac{1}{2}\sigma^2 - \frac{\lambda}{\rho+\lambda}(g_1 - g_0) - \frac{\rho\lambda}{\rho+\lambda}\left(\log\rho - \log\left(\rho + g_0 - \frac{\sigma^2}{2}\right)\right) < 0. \tag{25}$$

The additional term that comes from distorted consumption in 0 is not very large, so quantitatively this is not a very significant additional term. Nevertheless, it is important to keep in mind that the possibility of a binding constraint in the bad state of the world has dynamic implications for the likelihood of a binding constraint in the good state. In addition, our example has assumed Brownian risk alone, which biases against a binding constraint. If we instead assumed a greater role for stationary risk, either by increasing $\sigma$ or re-introducing jump risk, the sustainability condition in equation (24) would bind with even smaller values of $\Delta$. For example, if we set $\sigma = 10\%$ in our previous example, with $\lambda = 10\%$, then equation (24) will bind whenever $\Delta > 2.25\%$. If current growth is $\approx 2\%$, this would imply negative growth in the downside-risk state, in which case the sustainability constraint would bind going forward.

In contrast to the case of stationary risk (whether Brownian or disaster volatility), long-run risk makes sustainability less likely in both the partial and the general equilibrium settings. It strengthens the importance of risk in the general equilibrium world, and it decreases the gap between the consumption-wealth ratio and the risk-free rate in the partial equilibrium framework.

# 4 Growth and financial depth

The main message of the general equilibrium model is that sustainability is fundamentally a question of economic growth and risk. In a sense, this should not be surprising given that these have always been among the driving forces in the literatures on development, growth, and finance. So far, however, we have modeled these as exogenous forces, effectively out of the hands of countries and international agreements. This section considers growth and financial development in more detail and shows how they can be incorporated directly into a general-equilibrium sustainability framework.

## 4.1 Endogenous growth and investment

There are a variety of ways that one could endogenize growth in the model, including solving a competitive economy with AK production technology in the spirit of Romer (1986). Such a model would deliver the result that growth depends on a function of investment and capital depreciation. For the purposes of analyzing the impact of investment decisions on the sustainability constraint, we make the simplifying assumption that output is produced just using capital, $K$, which has a replacement cost of $\Phi(\iota)K$ (see Brunnermeier and Sannikov, 2014, 2016). In this setting, one can show that the growth rate is endogenous and given by:

$$g(\iota) = \Phi(\iota) - \delta,$$

where $\iota$ is the investment rate. In the presence of adjustment costs, $\Phi(.)$ would be both increasing and concave. Optimal investment satisfies Tobin's q:

$$Q\Phi'(\iota) = 1.$$

It is straightforward to show that the dividend yield in this setting equals $\rho$. Each unit of capital therefore generates dividends equal to $\rho Q$, and this must equal the ratio of consumption to capital,

$c/K = 1 - \iota$. Thus,

$$\rho Q = 1 - \iota.$$

Excess returns on the tree are given by the sum of the dividend yield and growth less the risk-free rate:

$$\frac{1-\iota}{Q} + g - r,$$

where the dividend yield reflects the investment rate.

We can solve in closed form by assuming the following functional form for the investment function. Let $\Phi(\iota) = \frac{1}{\phi}\log(1+\phi\iota)$. Then $\Phi' = 1/(1+\phi\iota)$, and optimal investment is given by

$$Q = 1 + \phi\iota \implies \iota = \frac{Q-1}{\phi}.$$

From market clearing we have $\rho Q = 1 - \iota$, and therefore

$$Q = \frac{1+\phi}{1+\rho\phi}.$$

Hence, we have $\iota = (\rho - 1)/(1+\rho\phi)$, and the growth rate is given by

$$g = \frac{1}{\phi}\log\left(\frac{1+\phi}{1+\rho\phi}\right) - \delta.$$

While this is just one possible specification of endogenous growth, it illustrates a potential policy response to a binding sustainability constraint. If we consider a broader measure of investment that includes physical capital and human capital, we have a policy prescription that aligns with the fundamentals of economic growth from Solow to Romer. One of the most effective ways to ensure that expected welfare growth for future generations remains positive is to continue to invest in physical capital, education, and R&D. But that is just one component of the sustainability condition in general equilibrium. The other is the magnitude of risk, whether in the form of Brownian volatility, rare disaster risk, or Markov regime switching.

### 4.2 Financial Depth and integration

So far, our models have assumed frictionless financial markets. In reality, countries have different levels of financial development, with some countries facing significant gaps in the return on saving and the cost of borrowed funds and others creating a wide span of assets available for domestic and global saving. Given the importance of financial risk in the general equilibrium sustainability model, it is worth exporing the roles of financial depth and financial integration.

### 4.2.1 Financial depth

There are a variety of ways to incorporate depth into the sustainability model. One way is to assume that countries differ in their ability to supply financial assets to savers, which has implications for the stability of consumption flows. Following Caballero, Farhi, and Gourinchas (2008), suppose that a country has "financial depth" $\kappa \in (0,1)$, which captures the fraction of future cash-flows that can be capitalized as assets. Financial depth, or stability, thus measures the suitable supply of assets available for savings.

Total output is given by $D_t$, a fraction $\kappa$ of which comes from a tree, and the remainder from endowments to new agents, which are saved until death. Agents are born and die at the same rate, $\rho$, and thus existing agents consume a fraction $\rho$ of their wealth. Critically, existing wealth does not include future endowments (those are the wealth of agents yet to enter). We modify equilibrium as follows.

First, market clearing for consumption is

$$\rho W = D$$

and the return on the tree satisfies

$$\frac{\kappa}{Q} + g - r = \sigma^2$$

where the $\kappa$ numerator reflects the smaller dividend. Market clearing for assets requires that all existing wealth is held in the tree. Hence $QD = W$, which along with the market clearing condition for consumption requires that $\rho Q = 1$. This means we have

$$\rho\kappa + g - r = \sigma^2 \implies r = \rho\kappa + g - \sigma^2 < \rho + g - \sigma^2.$$

As in Caballero, Farhi, and Gourinchas (2008), we have the result that lower financial depth reduces the risk-free rate. This has an immediate implication for the sustainability of consumption decisions. Namely, a closed economy with a limited ability to supply risk-sharing assets will be more likely to face a binding sustainability constraint. Recall that sustainability in general equilibrium requires that:

$$r - \rho + \sigma^2 - \frac{1}{2}\sigma^2 \geq 0.$$

Substituting the modified value of the risk-free rate, we have:

$$g - \frac{1}{2}\sigma^2 - \rho(1 - \kappa) \geq 0,$$

requiring

$$g \geq \frac{1}{2}\sigma^2 + \rho(1 - \kappa).$$

Our previous general equilibrium condition is equivalent to assuming that $\kappa = 1$. Economies with limited abilities to generate risk-sharing assets will therefore face more binding sustainability constraints. But this also provides a potential policy vehicle for improving sustainability. In addition to increasing growth through investments in physical and human capital, countries can also relax their sustainability constraints through increasing the depth of financial markets. So far, we have considered a closed-economy perspective. The impact of financial depth, however, likely depends on the level of financial integration with the rest of the world.

#### 4.2.2 Financial Integration

A simple way to see this point is to consider two countries differing in both growth and financial depth, $g$ and $\kappa$. As we have seen, the autarkic interest rates would be

$$r_1 = \rho\kappa_1 + g_1 - \sigma^2, \quad r_2 = \rho\kappa_2 + g_2 - \sigma^2.$$

As an important illustrative example, we could let 1 denote US and 2 denote China. Suppose that $r_1 > r_2$ with $g_2 \geq g_1$, which reflects the fact that China's growth rate has been well above that in the US, while its financial depth has been notably lower. The result, not surprisingly, is a flow of capital from China to the US, which we have seen in the data. As a simplification, assume that China and the US are equal sizes and make up the entire world. Then we have a world interest rate

$$r = \rho\bar{\kappa} + \bar{g} - \sigma^2,$$

where bars denote averages. In this case, the sustainability constraint requires

$$\bar{g} \geq \frac{1}{2}\sigma^2 + \rho(1 - \bar{\kappa}).$$

Financial integration mitigates the impact of limited financial depth on China's sustainability constraint, while it reduces the benefit of higher financial depth in the US.

# 5 Policies to address sustainability

Regardless of the specification, the general equilibrium framework provides a clear message about policies aimed to improve sustainability: countries can take measures to improve growth or to reduce the volatility of output through mitigating catastrophic risk, reducing financial frictions, or developing financial markets.

## 5.1 Mitigating catastrophic risk

We have already seen that catastrophic risk has the potential to overwhelm the importance of growth and Brownian risk in the sustainability condition. While the model assumes an exogenous process for disaster risk, both in terms of the frequency and severity of events, the process can be influenced by policies aimed at lowering the likelihood of catastrophic outcomes and mitigating the consequences of events that do happen. In the language of the model, we are looking for policies that lower the arrival rate, $\lambda$, of events, policies that alter the power distribution parameter, $\beta$, and policies that change the fraction of capital destroyed for a given shock $X$.

While there is no shortage of catastrophic risk facing the world, perhaps the two most salient examples are climate change and nuclear weapons. In both of these cases, countries have a limited ability to control the parameters governing the stochastic process alone, but more scope to develop policies aimed at mitigating the impact of shocks, particularly in the case of climate change. Thus, we should expect to see a combined emphasis on treaty-based solutions, on the one hand, and investments focused on mitigation, on the other.

The Netherlands pose an extreme example in the case of climate change. The low-lying nature of the country and abundance of waterways exposes the Dutch to extreme and rising risks associated with flooding. The Netherlands have invested in a combination of infrastructure and management strategies to mitigate the impact of flood risk, including dike renovation, floating homes, and modernized agriculture. At the same time, they also passed the Climate Act in 2019, which commits to a 95% reduction in greenhouse gasses by 2050. The mitigation strategies directly address the impact of the highest-risk climate shocks, while the climate agreement aligns Dutch policy with coordinated targets in Europe.

The Climate Act underscores a missing element in our general equilibrium model of sustainability: the role of externalities and the need for coordinated responses. Most catastrophic risks have both causes and consequences that cross international borders. The standard approach to modeling externalities at the global level has been the use of Integrated Assessment Models (IAMs), which combine a model of economic growth with an environmental externality that can be influenced by decisions about production and technology.[5] While such a model is beyond the scope of our sustainability framework, it is worth noting that the presence of externalities would almost surely change some of the model's conclusions. For example, in both Campbell and Martin (2022) and our own framework, the sustainability constraint does not change the optimal allocation between the safe and the risky asset. This may not be true in the presence of externalities, which might make the risky investment option less attractive in the presence of a constraint. Furthermore, the presence of externalities also places a premium on coordinated solutions to climate change, including William Nordhaus's "Climate Club" proposal (Nordhaus, 2015), international climate agreements (Harstad, 2016), and technology-sharing and trade policies (Weber and Peters, 2009).

### 5.2 Technology, investment, and human capital

In a sense, it is inevitable that a model of sustainable growth brings us back to the seminal contributions of Solow (1956), Romer (1990), and Mankiw et al. (1992), which place technology, investment, and human capital at the center of economic growth. Along with direct measures to avoid or mitigate the risks associated with climate change, these factors will continue to play a fundamental role in determining whether expected welfare for future generations increases or decreases—both at the country level, and for the world as a whole. In addition, these same factors are essential for building the capacity to directly address challenges posed by climate change, demographic change, and persistent warfare.

These factors can also play a critical role in reducing the probability and severity of catastrophic events. Transitioning to clean energy, reducing the likelihood of another Fukushima, or

[5] See Nordhaus (2017) for a discussion and comparison of various IAMs.

securing stockpiles of nuclear weapons will all require strategic investments in technology, infrastructure, and education. While our model does not allow for explicit decisions about investing in, for example, clean vs. dirty technologies, it is easy to imagine how directed technical change (Acemoglu et al., 2012, 2016) might provide an additional lever to reach or maintain a sustainable trajectory. Similarly, human capital can both lead to an increase in long-run growth (Barro, 2001; Jones and Romer, 2010), which relaxes the sustainability constraint, as well as a reduction in energy consumption (Shahbaz et al., 2022), which has a direct effect on the sustainability of current decisions about spending and asset allocation.

### 5.3 Financial development and integration

The policies above focus on reducing catastrophic risk and increasing the rate of economic growth, both of which tend to move an economy further above its sustainability constraint. Financial markets offer up the possibility of improving the management of risks, increasing the set of productive investments, and raising the return on wealth. The extensions in Section 4.2 showed some specific ways one might introduce financial depth and integration into the sustainability framework, but there are other possibilities.

The notion that we need more financial innovation is often met with skepticism, particularly after the adventures in securitization that preceeded the global financial crisis of 2008. There is little doubt, however, that financial development is crucial for economic growth (Levine, 1997). As Levine (1997) notes, "Theory suggests that financial instruments, markets, and institutions arise to mitigate the effects of information and transaction costs. Furthermore, a growing literature shows that differences in how well financial systems reduce information and transaction costs influence saving rates, investment decisions, technological innovation, and long-run growth rates..." In the language of our framework, financial development can increase growth and reduce the severity of risk. While it is challenging to disentangle the causal relationship between financial development and these outcomes, it seems clear that higher levels of financial development can help countries attain or remain on a sustainable path of growth.

As we have seen in Section 4.2, the lessons for financial integration are less clear. In the model setting, the impact of integration depends on the level of economic development of the respective countries. In practice, there is mixed evidence on the tendency for financial integration to reduce volatility. Kose et al. (2003), for example, find that the volatility of consumption increaased relative to the volatility of income for more financially integrated countries during the 1990s, a period of increased integration. And to the extent that financial integration contributes positively to growth and reduced volatility, this may be due more to indirect channels, such as the impact of openness on institutions and governness, than the direct effects on saving and diversification (Kose et al., 2009).

We have chosen to highlight financial approaches to reducing volatility, but the potential policy space is much richer than that. The connection between economic development and volatility is complex and involves the sectoral composition of production (Koren and Tenreyro, 2007), institutions (Acemoglu et al., 2003), and trade openness (Giovanni and Levchenko, 2009), among other factors. Any policies or institutions that reduce the volatility of consumption will help push countries further above their sustainability constraints.

## 6 Conclusion

What does it mean for an economy to be "sustainable" in general equilibrium? Defining sustainability as a non-decreasing path of expected future welfare, this paper argues that sustainability depends fundamentally on the forces of economic growth and volatility. While this may not seem surprising, in light of the central role these factors have played in macroeconomics, development, trade, and finance, it is not the result that one obtains from a partial-equilibrium framework. There, as Campbell and Martin (2022) have shown, risk allows the sustainable ratio of consumption to wealth to be higher than it would be in a world without risk. Effectively, the positive effect of the risk premium on future wealth more than offsets the negative effect of volatility on welfare due to risk aversion. An important exception to this occurs in the case of non-stationary risk, such as long-run risks that could manifest in persistently or permanently lower growth rates. In both partial and general equilibrium settings, non-stationary risks make it more likely that a sustainability constraint binds. Our results highlight the importance of understanding precisely the nature of risks facing the global economy.

In general equilibrium, expected returns are determined by the economic growth rate, and the risk-free rate adjusts to absorb the risk premium. Here, stationary risk only affects sustainability through its impact on the concavity of utility, which decreases the sustainable consumption-wealth ratio, and the only way to increase the expected return to saving is through increasing economic growth. Non-stationary risk makes the downside state of the world more likely, which further reduces expected welfare. In a sense, general equilibrium returns the conversation to the same factors that have been fundamental to growth and development all along: investment, education, technology, and financial depth and integration.

This, then, raises the question of which framework most accurately captures the tradeoffs between growth and sustainability. The answer likely depends on the degree of aggregation: whether we are considering the world as a whole, an individual country, or an SOE or institution within a country. The smaller the unit of aggregation, the more reasonable it is that the presence of a risky investment opportunity relaxes a sustainability constraint. This would be true for individuals, institutional endowments, pension funds, SOEs, and perhaps even for some sovereign wealth funds. One of the main lessons of Campbell and Martin (2022) is that, for these agents, sustainability considerations do not distort the allocation of capital (between safe and risky investments) and so, subject to standard externalities, market returns can and should serve as signals for how to invest capital. Nonetheless, even for these countries long-run risks would pose a challenge for sustainability. At the highest levels of aggregation, however, general equilibrium may be a more appropriate framework. The world economy as a whole cannot escape the fundamental forces of growth and volatility.

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# Appendices

## A Derivations and proofs for Section 2

**Partial Equilibrium** We first solve the standard optimization problem. The HJB is

$$\rho V = \max_{c,\alpha} \left\{ \log(c) + V'(r + \alpha\mu - c/W)W + \alpha^2 \frac{\sigma^2}{2} W^2 V'' \right\}.$$

With log utility the value function can be written as $V = \frac{\log W}{\rho} + b$, so that $V' = \frac{1}{\rho W}$ and $V'' = -\frac{1}{\rho W^2}$. Plugging in we have

$$\rho V = \max_{c,\alpha} \left\{ \log(c) + \frac{1}{\rho}(r + \alpha\mu - c/W) - \alpha^2 \frac{\sigma^2}{2\rho} \right\},$$

and taking first-order conditions we have the standard conditions

$$\frac{1}{c} = \frac{1}{W\rho} \implies c = \rho W, \quad \frac{1}{\rho}\mu = \alpha\frac{\sigma^2}{\rho} \implies \alpha = \frac{\mu}{\sigma^2}. \tag{26}$$

The value function therefore satisfies

$$\begin{aligned} \log W + \rho b &= \log(\rho) + \log W + \frac{1}{\rho}(r + SR^2 - \rho) - \frac{SR^2}{2\rho}, \\ \rho b &= \log(\rho) + \frac{1}{\rho}\left(r - \rho + \frac{SR^2}{2}\right). \end{aligned}$$

Hence, we have

$$V = \frac{\log W}{\rho} + \frac{\log(\rho)}{\rho} + \frac{1}{\rho^2}\left(r - \rho + \frac{SR^2}{2}\right).$$

When the sustainability constraint binds, we then have the following condition on $\theta, \alpha$:

$$r + \alpha\mu - \theta - \frac{1}{2}\alpha^2\sigma^2 = 0 \implies \nu = 0. \tag{27}$$

The sustainability constraint imposes $\nu = 0$. We can plug into equation (35) to write $b$ as

$$\rho b = \log(\theta) + \frac{1}{\rho}\nu = \log(\theta).$$

Maximizing welfare comes down to maximizing $\theta$ subject to the sustainability constraint. This is why the optimal portfolio decision is the same as before. The constraint does not distort the risk-reward tradeoff from investment, so the optimal allocation provides the highest possible consumption rate. We show this now. For a given consumption and portfolio choice $\theta, \alpha$, welfare is given by

$$V = \frac{\log W}{\rho} + \frac{\log(\theta)}{\rho} + \frac{1}{\rho^2}\left(r - \theta + \alpha\mu - \frac{1}{2}\alpha^2\sigma^2\right). \tag{28}$$

We can write the HJB:

$$\rho A \log W + \rho b = \log(\theta) + \log W + A(r + \alpha\mu - \theta) - \alpha^2\frac{\sigma^2}{2}A.$$

We immediately have $\rho A = 1$ as before. Continuing

$$\rho b = \log(\theta) + \frac{1}{\rho}\left(r + \alpha\mu - \theta - \alpha^2\frac{\sigma^2}{2}\right).$$

Note that maximizing welfare at this stage corresponds to choosing $\theta, \alpha$ to maximize $b$. Doing so, we are guaranteed to get the same FOC as before $\theta = \rho$ and $\alpha = \frac{\mu}{\sigma^2}$. Again it becomes clear that maximizing welfare with a binding constraint comes down to maximizing $\theta$ subject to the sustainability constraint. This is why the optimal portfolio decision is the same as before. The constraint does not distort the risk-reward tradeoff from investment, so the optimal allocation provides the highest possible consumption rate.

**CRRA utility with $\gamma > 1$** With CRRA utility there are a few changes. First, the welfare function is now $V = AW^{1-\gamma}$. The optimality conditions are now

$$\alpha = \frac{\mu}{\gamma\sigma^2}$$

and

$$\begin{aligned}\frac{c}{W} &= \frac{\rho + (\gamma - 1)(r + \alpha\mu - \frac{1}{2}\gamma\alpha^2\sigma^2)}{\gamma}, \\ &= \frac{\rho + (\gamma - 1)(r + \frac{1}{2\gamma}SR^2)}{\gamma}.\end{aligned}$$

Given the welfare function, the natural consideration for sustainability is $\mathbb{E}[\mathrm{d}V/V]$. By Itô's Lemma, the sustainability constraint is modified to reflect risk-aversion (see Campbell and Martin (2022)):

$$\mathbb{E}\left[\mathrm{d}V/V\right] \propto r - \theta + \alpha\mu - \gamma\frac{1}{2}\alpha^2\sigma^2. \tag{29}$$

Note that the portfolio terms $r + \alpha\mu - \gamma\frac{1}{2}\alpha^2\sigma^2$ are in the numerator of consumption, and hence can be written as $r + \frac{1}{2\gamma}SR^2 - \theta$. This also means that when we plug in for consumption, the terms will combine nicely.

**Proposition 7.** *With CRRA utility, the sustainability constraint is given by*

$$r - \rho + \frac{SR^2}{2\gamma} \geq 0. \tag{30}$$

*Proof.* At the optimal, the sustainability constraint can be written

$$\begin{aligned}
\mathbb{E}[\mathrm{d}V/V] \propto r + \frac{1}{2\gamma}SR^2 - \frac{\rho + (\gamma-1)(r + \frac{1}{2\gamma}SR^2)}{\gamma}, \\
= \frac{\gamma}{\gamma}\left(r + \frac{1}{2\gamma}SR^2\right) - \frac{\rho}{\gamma} - \frac{\gamma-1}{\gamma}\left(r + \frac{1}{2\gamma}SR^2\right), \\
= \frac{1}{\gamma}\left(r + \frac{1}{2\gamma}SR^2\right) - \frac{\rho}{\gamma}, \\
= \frac{1}{\gamma}\left(r - \rho + \frac{SR^2}{2\gamma}\right).
\end{aligned}$$

Since $\gamma > 0$, this yields the result. □

**Recursive Preferences** We can also solve with recursive Epstein-Zin (EZ) preferences with EIS $\epsilon$.

**Proposition 8.** *With recursive (Epstein-Zin) utility, optimal consumption is given by*

$$c/W = \epsilon\rho + (1-\epsilon)\left(r + \frac{1}{2}\gamma\alpha^2\sigma^2\right) = \epsilon\rho + (1-\epsilon)\left(r + \frac{1}{2\gamma}SR^2\right)$$

*and the sustainability constraint is given by*

$$r - \rho + \frac{SR^2}{2\gamma} \geq 0. \tag{31}$$

*Proof.* Consumption is standard and follows from plugging in $\alpha = \mu/(\gamma\sigma^2)$. At the optimal, the sustainability constraint can be written

$$\begin{aligned}
\mathbb{E}[\mathrm{d}V/V] &\propto r + \alpha\mu - \gamma\frac{1}{2}\alpha^2\sigma^2 - \left(\epsilon\rho + (1-\epsilon)\,(r + \frac{1}{2}\gamma\alpha^2\sigma^2)\right), \\
&= r + \frac{1}{2}\gamma\alpha^2\sigma^2 - \left(\epsilon\rho + (1-\epsilon)\,(r + \frac{1}{2}\gamma\alpha^2\sigma^2)\right), \\
&= \epsilon\left(r + \frac{1}{2}\gamma\alpha^2\sigma^2\right) - \epsilon\rho, \\
&= \epsilon\left(r - \rho + \frac{SR^2}{2\gamma}\right),
\end{aligned}$$

and the result follows since $\epsilon > 0$. □

Hence, the condition for a binding sustainability constraint is the same with CRRA or EZ preferences.

**General Equilibrium with Recursive Preferences** With recursive preferences, the risk-free rate satisfies

$$r = \rho + \frac{1}{\epsilon}\mu_c - \frac{\gamma\left(1 + \frac{1}{\epsilon}\right)}{2}\sigma_c^2,$$

where $\mu_c$ and $\sigma_c$ are the expected growth rate and volatility of consumption. The excess risk-premium equals $\gamma\sigma^2$ which means $SR = \gamma\sigma$. Note that we do not actually need to use market clearing for consumption to pin down the dividend yield; all we need is the excess return, which is pinned down by risk-aversion. We already have optimal consumption as a function of returns. Plugging these into the sustainability constraint we have

$$\begin{aligned}
&\epsilon\left(\rho + \frac{1}{\epsilon}g - \frac{\gamma\left(1+\frac{1}{\epsilon}\right)}{2}\sigma^2 - \rho + \frac{\gamma^2\sigma^2}{2\gamma}\right) \geq 0, \\
&\epsilon\left(\frac{1}{\epsilon}g - \frac{\gamma\left(1+\frac{1}{\epsilon}\right)}{2}\sigma^2 + \frac{\gamma\sigma^2}{2}\right) \geq 0, \\
&\epsilon\left(\frac{1}{\epsilon}g + \frac{\gamma\sigma^2}{2}\left(1 - 1 - \frac{1}{\epsilon}\right)\right) \geq 0, \\
&\epsilon\left(\frac{1}{\epsilon}g - \frac{\gamma\sigma^2}{2}\left(\frac{1}{\epsilon}\right)\right) \geq 0, \\
&g - \frac{\gamma\sigma^2}{2} \geq 0.
\end{aligned}$$

## B Proof of Proposition 2, Disaster Risk in Section 3.1

From Campbell and Martin (2022), optimal consumption for CRRA utility with disaster risk is

$$\frac{c}{W} = \frac{\rho + (\gamma - 1)(r + \hat{\mu} - \frac{1}{2}\gamma\sigma^2) - \lambda \mathbb{E}\left[X^{1-\gamma} - 1\right]}{\gamma}.$$

and Pindyck and Wang (2013) have

$$\frac{c}{W} = \rho + (\gamma - 1)\left(g - \frac{1}{2}\gamma\sigma^2 - \frac{\lambda}{1-\gamma}\mathbb{E}\left[1 - X^{1-\gamma}\right]\right).$$

The sustainability constraint (SC) with $\alpha = 1$ is

$$SC = r - c/w + \hat{\mu} - \frac{1}{2}\gamma\sigma^2 + \frac{\lambda}{1-\gamma}\mathbb{E}\left[X^{1-\gamma} - 1\right] \geq 0.$$

Let's break these in pieces. First, we have

$$\begin{aligned} r - c/w = \rho + \gamma g - \frac{\gamma(1+\gamma)}{2}\sigma^2 - \lambda \mathbb{E}\left[(X^{-\gamma} - 1)\right] \\ - \left(\rho + (\gamma - 1)\left(g - \frac{1}{2}\gamma\sigma^2 - \frac{\lambda}{1-\gamma}\mathbb{E}\left[1 - X^{1-\gamma}\right]\right)\right), \end{aligned}$$

and so

$$\begin{aligned} r - c/w &= g - \frac{\gamma(1+\gamma)}{2}\sigma^2 - \lambda \mathbb{E}\left[(X^{-\gamma} - 1)\right] - \left((\gamma - 1)\left(-\frac{1}{2}\gamma\sigma^2 - \frac{\lambda}{1-\gamma}\mathbb{E}\left[1 - X^{1-\gamma}\right]\right)\right), \\ &= g - \frac{\gamma(1+\gamma)}{2}\sigma^2 - \lambda \mathbb{E}\left[(X^{-\gamma} - 1)\right] + \frac{1}{2}(\gamma - 1)\gamma\sigma^2 + \lambda \mathbb{E}\left[1 - X^{1-\gamma}\right], \\ &= g - \frac{\gamma}{2}\sigma^2(1 + \gamma - (\gamma - 1)) - \lambda \mathbb{E}\left[(X^{-\gamma} - 1)\right] + \lambda \mathbb{E}\left[1 - X^{1-\gamma}\right], \\ &= g - \gamma\sigma^2 - \lambda \mathbb{E}\left[(X^{-\gamma} - 1)\right] + \lambda \mathbb{E}\left[1 - X^{1-\gamma}\right]. \end{aligned}$$

Then we can write $\hat{\mu} - \frac{1}{2}\gamma\sigma^2$ as

$$\begin{aligned} \hat{\mu} - \frac{1}{2}\gamma\sigma^2 &= \gamma\sigma^2 + \lambda \mathbb{E}\left[(1 - X)(X^{-\gamma} - 1)\right] + \lambda \mathbb{E}\left[(1 - X)\right] - \frac{1}{2}\gamma\sigma^2, \\ &= \frac{1}{2}\gamma\sigma^2 + \lambda \mathbb{E}\left[(1 - X)X^{-\gamma}\right]. \end{aligned}$$

Combining these we have

$$
\begin{aligned}
r - c/w + \hat{\mu} - \frac{1}{2}\gamma\sigma^2 &= g - \gamma\sigma^2 - \lambda\mathbb{E}\left[(X^{-\gamma} - 1)\right] + \lambda\mathbb{E}\left[1 - X^{1-\gamma}\right] + \frac{1}{2}\gamma\sigma^2 + \lambda\mathbb{E}\left[(1-X)X^{-\gamma}\right], \\
&= g - \frac{1}{2}\gamma\sigma^2 + \lambda\mathbb{E}\left[1 - X^{1-\gamma} + (1-X)X^{-\gamma} - (X^{-\gamma} - 1)\right], \\
&= g - \frac{1}{2}\gamma\sigma^2 + \lambda\mathbb{E}\left[1 - X^{1-\gamma} + X^{-\gamma} - X^{1-\gamma} - X^{-\gamma} + 1\right], \\
&= g - \frac{1}{2}\gamma\sigma^2 + 2\lambda\mathbb{E}\left[1 - X^{1-\gamma}\right].
\end{aligned}
$$

Finally,

$$
\begin{aligned}
SC &= g - \frac{1}{2}\gamma\sigma^2 + 2\lambda\mathbb{E}\left[1 - X^{1-\gamma}\right] + \frac{\lambda}{1-\gamma}\mathbb{E}\left[X^{1-\gamma} - 1\right], \\
&= g - \frac{1}{2}\gamma\sigma^2 + \lambda\mathbb{E}\left[1 - X^{1-\gamma}\right]\left(2 - \frac{1}{1-\gamma}\right), \\
&= g - \frac{1}{2}\gamma\sigma^2 + \lambda\mathbb{E}\left[1 - X^{1-\gamma}\right]\left(\frac{1-2\gamma}{1-\gamma}\right).
\end{aligned}
$$

Let $X$ have Power distribution in $(0,1)$ with shape $\beta > 0$. Then $\mathbb{E}\left[X^n\right] = \frac{\beta}{\beta+n}$ and so

$$
\mathbb{E}\left[X^{1-\gamma}\right] = \frac{\beta}{\beta + 1 - \gamma}
$$

which means

$$
\mathbb{E}\left[1 - X^{1-\gamma}\right] = \frac{1-\gamma}{\beta + 1 - \gamma}
$$

and therefore we have

$$
\begin{aligned}
SC &= g - \frac{1}{2}\gamma\sigma^2 + \lambda\frac{1-2\gamma}{\beta+1-\gamma}, \\
&= g - \frac{1}{2}\gamma\sigma^2 - \lambda\frac{2\gamma-1}{\beta-(\gamma-1)}.
\end{aligned}
$$

# C Results for Long-run risk in Section 3.2,

We first do partial equilibrium with log utility. There are two states, $0, 1$, with $\mu_0 < \mu_1$. The state changes at Poisson rate $\lambda$. Hence, state 1 is the good state.

**Partial Equilibrium** We first solve the standard optimization problem. Let the value function in state $i$ be $V_i$. Let $\alpha_i$ denote the portfolio and let $c_i$ denote consumption. The investor HJB is

$$\rho V_i = \max_{c,\alpha} \left\{ \log(c) + V_i'(r + \alpha\mu_i - c/W)W + \alpha_i^2 \frac{\sigma^2}{2} W^2 V_i'' + \lambda \left(V_j - V_i\right) \right\}.$$

**Proposition 9.** *Optimality conditions are $c_i = \rho W$ and $\alpha_i = \frac{\mu_i}{\sigma^2}$. Welfare is given by*

$$V_i = \frac{\log W}{\rho} + \frac{\log(\rho)}{\rho} + \frac{1}{\rho^2}\left(r - \rho + \frac{SR_i^2}{2}\right) - \frac{\lambda}{2\rho^2(\rho + 2\lambda)}\left(SR_i^2 - SR_j^2\right). \tag{32}$$

*In particular, the Markov shock does not affect the optimality conditions but it does affect welfare.*

*Proof.* With log utility the value function can be written as

$$V_i = \frac{\log W}{\rho} + b_i,$$

so that $V' = \frac{1}{\rho W}$ and $V'' = -\frac{1}{\rho W^2}$. Plugging in we have

$$\rho V_i = \max_{c,\alpha} \left\{ \log(c) + \frac{1}{\rho}(r + \alpha\mu_i - c/W) - \alpha_i^2 \frac{\sigma^2}{2\rho} + \lambda \left(b_j - b_i\right) \right\},$$

and taking FOCs we have the standard conditions

$$\frac{1}{c} = \frac{1}{W\rho} \implies c = \rho W, \quad \frac{1}{\rho}\mu_i = \alpha_i \frac{\sigma^2}{\rho} \implies \alpha = \frac{\mu_i}{\sigma^2}. \tag{33}$$

Importantly, the Markov shock does not affect these optimality conditions. The value function therefore satisfies

$$\begin{aligned}
\log W + \rho b_i &= \log(\rho) + \log W + \frac{1}{\rho}(r + SR_i^2 - \rho) - \frac{SR_i^2}{2\rho} + \lambda\left(b_j - b_i\right), \\
\rho b_i &= \log(\rho) + \frac{1}{\rho}(r - \rho) + \frac{SR_i^2}{2\rho} + \lambda\left(b_j - b_i\right), \\
(\rho + \lambda) b_i &= \log(\rho) + \frac{1}{\rho}(r - \rho) + \frac{SR_i^2}{2\rho} + \lambda b_j, \\
(\rho + \lambda) b_j &= \log(\rho) + \frac{1}{\rho}(r - \rho) + \frac{SR_j^2}{2\rho} + \lambda b_i.
\end{aligned}$$

Subtracting equations for $i, j$ we therefore have

$$
\begin{aligned}
(\rho+\lambda)(b_i-b_j) &= \frac{SR_i^2-SR_j^2}{2\rho}-\lambda(b_i-b_j),\\
(\rho+2\lambda)(b_i-b_j) &= \frac{SR_i^2-SR_j^2}{2\rho},\\
(b_i-b_j) &= \frac{SR_i^2-SR_j^2}{2\rho(\rho+2\lambda)}.
\end{aligned}
$$

Hence, we have

$$
\rho b_i = \log(\rho)+\frac{1}{\rho}\left(r-\rho+\frac{SR_i^2}{2}\right)-\frac{\lambda}{2\rho(\rho+2\lambda)}\left(SR_i^2-SR_j^2\right).
$$

□

**General Welfare Function** We now characterize welfare for given $\theta_i, \alpha_i$. Recall that $\nu = r+\alpha\mu-\theta-\alpha^2\frac{\sigma^2}{2}$.

**Proposition 10.** *For a given $\theta, \alpha$, welfare is given by $V_i = \frac{\log W}{\rho}+b_i$, with*

$$
\rho b_i = \log(\theta_i)+\frac{1}{\rho}\nu_i-\frac{\lambda}{\rho+2\lambda}\left(\log(\theta_i)-\log(\theta_j)+\frac{1}{\rho}\left(\nu_i-\nu_j\right)\right). \tag{34}
$$

*Proof.* Let the welfare function be given by

$$
V_i = A_i\log W+b_i,
$$

so that the first coefficient could change with the state. Then we have

$$
\rho A_i\log W+\rho b_i = \log(\theta_i)+\log W+A_i\left(r+\alpha_i\mu_i-\theta_i-\alpha_i^2\frac{\sigma^2}{2}\right)+\lambda\left(b_j-b_i\right).
$$

We immediately have $\rho A_i = 1$ as before. Continuing

$$
\begin{aligned}
\rho b_i &= \log(\theta_i)+\frac{1}{\rho}\left(r+\alpha_i\mu_i-\theta_i-\alpha_i^2\frac{\sigma^2}{2}\right)+\lambda\left(b_j-b_i\right),\\
(\rho+\lambda)b_i &= \log(\theta_i)+\frac{1}{\rho}\left(r+\alpha_i\mu_i-\theta_i-\alpha_i^2\frac{\sigma^2}{2}\right)+\lambda b_j,
\end{aligned}
$$

and we also have

$$(\rho+\lambda)b_j = \log(\theta_j) + \frac{1}{\rho}\left(r + \alpha_j\mu_j - \theta_j - \alpha_j^2\frac{\sigma^2}{2}\right) + \lambda b_i.$$

Plugging in for $\nu_i$ we can write

$$\rho b_i = \log(\theta_i) + \frac{1}{\rho}\nu_i - \lambda\left(b_i - b_j\right). \tag{35}$$

We can combine to get

$$\begin{aligned}(\rho+\lambda)(b_i-b_j) &= \log(\theta_i) - \log(\theta_j) + \frac{1}{\rho}(\nu_i-\nu_j) - \lambda(b_i-b_j),\\ (\rho+2\lambda)(b_i-b_j) &= \log(\theta_i) - \log(\theta_j) + \frac{1}{\rho}(\nu_i-\nu_j).\end{aligned}$$

Hence,

$$(b_i-b_j) = \frac{1}{\rho+2\lambda}\left(\log(\theta_i) - \log(\theta_j) + \frac{1}{\rho}\left(\nu_i-\nu_j\right)\right). \tag{36}$$

Plugging in we therefore have

$$\begin{aligned}\rho b_i &= \log(\theta_i) + \frac{1}{\rho}\left(\nu_i\right) - \frac{\lambda}{\rho+2\lambda}\left(\log(\theta_i) - \log(\theta_j) + \frac{1}{\rho}\left(\nu_i-\nu_j\right)\right),\\ &= \left(\frac{\rho+\lambda}{\rho+2\lambda}\right)\left(\log(\theta_i) + \frac{1}{\rho}\left(\nu_i\right)\right) + \frac{\lambda}{\rho+2\lambda}\left(\log(\theta_j) + \frac{1}{\rho}\left(\nu_j\right)\right).\end{aligned}$$

At the optimal we have

$$\nu_i = r - \rho + \frac{1}{2}SR_i^2.$$

□

**Adding Sustainability Constraint** We now impose that welfare cannot have a negative drift. The expected change in welfare is the drift in wealth minus the Itô term reflecting risk plus the Poisson term.

$$\begin{aligned}\mathbb{E}[\mathrm{d}V_i] &= \frac{1}{\rho}\left(r + \alpha_i\mu_i - \theta_i - \frac{1}{2}\alpha_i^2\sigma^2\right) - \lambda(b_i-b_j),\\ &= \frac{\nu_i}{\rho} - \lambda(b_i-b_j).\end{aligned} \tag{37}$$

*Proof of Proposition 3.* Let's first go directly to the optimized conditions, in which case $\theta_i = \theta_j = \rho$

and $\nu_i - \nu_j = \frac{1}{2}(SR_i^2 - SR_j^2)$. Using equation (21), we can write

$$\mathbb{E}[\mathrm{d}V_i] = r - \rho + \frac{1}{2}SR_i^2 - \frac{\lambda}{2(\rho+2\lambda)}\left(SR_i^2 - SR_j^2\right).$$

□

When the constraint binds, we then have the following condition on $\theta_i, \alpha_i$:

$$r + \alpha_i \mu_i - \theta_i - \frac{1}{2}\alpha_i^2\sigma^2 = \rho\lambda(b_i - b_j), \implies \nu_i = \rho\lambda(b_i - b_j). \tag{38}$$

Because $b_i - b_j$ is a function of $\theta_i, \alpha_i$ from (21), we cannot solve for $\theta_i$ explicitly.

**Proposition 11.** *When the SC binds, the optimal portfolio is unchanged, $\alpha_i = \frac{\mu_i}{\sigma^2}$, and consumption solves*

$$\theta_i + \frac{\rho\lambda}{\rho+\lambda}\log(\theta_i) = r + \frac{1}{2}SR_i^2 + \frac{\rho\lambda}{\rho+\lambda}\left(\log(\rho) + (r - \rho + \frac{1}{2}SR_j^2)/\rho\right). \tag{39}$$

*This means that $\theta < \rho$, i.e., we need a lower $\theta$ to satisfy the constraint.*

*Proof.* By solving for $b_i - b_j$ from the sustainability constraint, we can plug into equation (35) to write $b_i$ as

$$\rho b_i = \log(\theta_i) + \frac{1}{\rho}\nu_i - \frac{1}{\rho}(\nu_i) = \log(\theta_i).$$

Maximizing welfare comes down to maximizing $\theta$ subject to the sustainability constraint. The difference is that our sustainability constraint is not so simple, as noted above. Use $\nu_i = \rho\lambda(b_i - b_j)$ from the sustainability constraint with equation (21):

$$\begin{aligned}
\nu_i &= \frac{\rho\lambda}{\rho+2\lambda}\left(\log(\theta_i) - \log(\theta_j) + \frac{1}{\rho}(\nu_i - \nu_j)\right),\\
\nu_i &= \frac{\rho\lambda}{\rho+2\lambda}\left(\log(\theta_i) - \log(\theta_j) - \nu_j/\rho\right) + \frac{\lambda}{\rho+2\lambda}\nu_i,\\
\nu_i\left(1 - \frac{\lambda}{\rho+2\lambda}\right) &= \frac{\rho\lambda}{\rho+2\lambda}\left(\log(\theta_i) - \log(\theta_j) - \nu_j/\rho\right),\\
\nu_i\left(\frac{\rho+2\lambda-\lambda}{\rho+2\lambda}\right) &= \frac{\rho\lambda}{\rho+2\lambda}\left(\log(\theta_i) - \log(\theta_j) - \nu_j/\rho\right),\\
\nu_i\left(\frac{\rho+\lambda}{\rho+2\lambda}\right) &= \frac{\rho\lambda}{\rho+2\lambda}\left(\log(\theta_i) - \log(\theta_j) - \nu_j/\rho\right),\\
\nu_i &= \frac{\rho\lambda}{\rho+\lambda}\left(\log(\theta_i) - \log(\theta_j) - \nu_j/\rho\right).
\end{aligned}$$

This means we have

$$r + \alpha_i \mu_i - \theta_i - \frac{1}{2}\alpha_i^2 \sigma^2 = \frac{\rho\lambda}{\rho + \lambda}\left(\log(\theta_i) - \log(\theta_j) - \nu_j/\rho\right),$$

or,

$$\theta_i + \frac{\rho\lambda}{\rho + \lambda}\log(\theta_i) = r + \alpha_i \mu_i - \frac{1}{2}\alpha_i^2 \sigma^2 + \frac{\rho\lambda}{\rho + \lambda}\left(\log(\theta_j) + \nu_j/\rho\right). \tag{40}$$

Recall that we want to maximize $\log\theta$ (i.e., maximize $\theta$) subject to the constraint. The LHS is increasing in $\theta$, which means we want to maximize the RHS. Hence,

$$\alpha_i = \frac{\mu_i}{\sigma^2},$$

the same FOC from earlier. The portfolio decision is not distorted, same as before. Thus, the consumption rate implicitly solves

$$\theta_i + \frac{\rho\lambda}{\rho + \lambda}\log(\theta_i) = r + \frac{1}{2}SR_i^2 + \frac{\rho\lambda}{\rho + \lambda}\left(\log(\rho) + (r - \rho + \frac{1}{2}SR_j^2)/\rho\right), \tag{41}$$

where have substituted in state $j$. □

**General Equilibrium**

*Proof of Proposition 4.* We need to characterize the wedge $\rho\lambda(b_i - b_j)$ in GE. From equation (21), we have

$$\rho\lambda(b_i - b_j) = \frac{\rho\lambda}{\rho + 2\lambda}\left(\log(\theta_i) - \log(\theta_j) + \frac{1}{\rho}\left(\nu_i - \nu_j\right)\right).$$

Thus, in GE and plugging in optimal consumption we have

$$\rho\lambda(b_i - b_j) = \frac{\lambda}{\rho + 2\lambda}\left(g_i - g_j\right).$$

Thus, sustainability will bind in 1 if $g_1 - \frac{1}{2}\sigma^2 - \frac{\lambda}{\rho+2\lambda}\left(g_1 - g_0\right) < 0$. □

Two things to note. First, if $g_0$ is sufficiently low, then the constraint will bind in 0, which means that optimal consumption will not be $\rho$—this will affect the constraint. Second, it's plausible that $\lambda$ is not very high, maybe there is a 2% chance of a disaster in a given year? The growth loss $g_1 - g_0$ has to be substantial for this to matter. With one-sided risk (the next section) the denominator will be $\rho + \lambda$ instead of $\rho + 2\lambda$ which will increase the cost, making the SC more likely to bind. It takes substantial growth risks to make this bind since $\lambda$ is not 1.

### C.1 Downside Risk: low growth absorbing state

Now suppose that we start in the good state and we switch to the bad state Poisson rate $\lambda$. But now we think of the bad state as permanent disaster (climate risk?) and so once we move to the bad state we stay there.

To start, let's suppose the SC constraint does NOT bind in the bad state so we can do a standard optimization. (The GE analysis might suggest otherwise.) The previous optimization goes through. We update the value functions as follows. We update the value functions as follows:

#### Partial Equilibrium

**Proposition 12.** *With downside (absorbing) risk, the state-*0 *welfare intercept is given simply by*

$$\rho b_0 = \log(\rho) + \frac{1}{\rho}(r-\rho) + \frac{SR_0^2}{2\rho},$$

*and the state-*1 *welfare intercept is*

$$\rho b_1 = \log(\rho) + \frac{1}{\rho}\left(r - \rho + \frac{SR_1^2}{2}\right) - \frac{\lambda}{2\rho(\rho+\lambda)}\left(SR_1^2 - SR_0^2\right).$$

*Proof.* First, the state-0 function follows immediately since there is no state risk. The state-1 function is

$$\begin{aligned}
\rho b_1 &= \log(\rho) + \frac{1}{\rho}\left(r - \rho + \frac{SR_1^2}{2}\right) - \lambda\left(b_1 - b_0\right), \\
(\rho+\lambda) b_1 &= \log(\rho) + \frac{1}{\rho}\left(r - \rho + \frac{SR_1^2}{2}\right) + \lambda b_0.
\end{aligned}$$

Plugging in we have:

$$
\begin{aligned}
(\rho+\lambda)b_1 &= \log(\rho) + \frac{1}{\rho}\left(r-\rho+\frac{SR_1^2}{2}\right) + \frac{\lambda}{\rho}\left(\log(\rho)+\frac{1}{\rho}(r-\rho)+\frac{SR_0^2}{2\rho}\right),\\
(\rho+\lambda)b_1 &= \log(\rho) + \frac{1}{\rho}\left(r-\rho+\frac{SR_1^2}{2}\right) + \frac{\lambda}{\rho}\left(\log(\rho)+\frac{1}{\rho}(r-\rho)+\frac{SR_0^2}{2\rho}\right),\\
(\rho+\lambda)b_1 &= \log(\rho)\left(1+\frac{\lambda}{\rho}\right) + \frac{1}{\rho}(r-\rho)\left(1+\frac{\lambda}{\rho}\right) + \frac{SR_1^2}{2\rho} + \frac{\lambda}{\rho}\left(\frac{SR_0^2}{2\rho}\right),\\
(\rho+\lambda)b_1 &= \log(\rho)\left(\frac{\rho+\lambda}{\rho}\right) + \frac{1}{\rho}(r-\rho)\left(\frac{\rho+\lambda}{\rho}\right) + \frac{SR_1^2}{2\rho} + \frac{\lambda}{\rho}\left(\frac{SR_0^2}{2\rho}\right),\\
\rho b_1 &= \log(\rho) + \frac{1}{\rho}(r-\rho) + \frac{SR_1^2}{2\rho}\left(\frac{\rho}{\rho+\lambda}\right) + \frac{\lambda}{\rho}\left(\frac{SR_0^2}{2\rho}\left(\frac{\rho}{\rho+\lambda}\right)\right).
\end{aligned}
$$

Let's rearrange this so it looks more like what we had earlier. We have

$$
\begin{aligned}
\rho b_1 &= \log(\rho) + \frac{1}{\rho}\left(r-\rho+\frac{SR_1^2}{2}\right) + \frac{SR_1^2}{2\rho}\left(\frac{\rho}{\rho+\lambda}-1\right) + \frac{\lambda}{\rho+\lambda}\left(\frac{SR_0^2}{2\rho}\right),\\
\rho b_1 &= \log(\rho) + \frac{1}{\rho}\left(r-\rho+\frac{SR_1^2}{2}\right) - \frac{SR_1^2}{2\rho}\left(\frac{\lambda}{\rho+\lambda}\right) + \frac{\lambda}{\rho+\lambda}\left(\frac{SR_0^2}{2\rho}\right),\\
\rho b_1 &= \log(\rho) + \frac{1}{\rho}\left(r-\rho+\frac{SR_1^2}{2}\right) - \frac{\lambda}{2\rho(\rho+\lambda)}\left(SR_1^2 - SR_0^2\right).
\end{aligned}
$$

Compared to earlier, the last term has $\rho+\lambda$ in the denominator instead of $\rho+2\lambda$. □

For later, it is convenient to write

$$
\begin{aligned}
\lambda\left(b_1-b_0\right) &= \log(\rho) + \frac{1}{\rho}\left(r-\rho+\frac{SR_1^2}{2}\right) - \rho b_1, && (42)\\
&= \frac{\lambda}{2\rho(\rho+\lambda)}\left(SR_1^2 - SR_0^2\right). && (43)
\end{aligned}
$$

**General Welfare Function** We now characterize welfare for given $\theta_i, \alpha_i$. Recall that $\nu = r + \alpha\mu - \theta - \alpha^2\frac{\sigma^2}{2}$.

**Proposition 13.** *With downside (absorbing) risk, for a given $\theta, \alpha$ welfare is given by $V_i = \frac{\log W}{\rho} + b_i$, with*

$$\rho b_0 = \log(\theta_0) + \frac{1}{\rho}\nu_0.$$

*(no long-run risk) and*

$$\rho b_1 = \log(\theta_1) + \frac{1}{\rho}\nu_1 - \frac{\lambda}{\rho+\lambda}\left(\log(\theta_1) - \log(\theta_0) + \frac{1}{\rho}\left(\nu_1-\nu_0\right)\right). \quad (44)$$

*Proof.* The welfare function in state 0 follows immediately from previous analysis. It remains to characterize the additional term in state 1. As we did earlier, we can write

$$\begin{aligned}
\rho b_1 &= \log(\theta_1) + \frac{1}{\rho}\nu_1 + \lambda\left(b_0 - b_1\right), \\
(\rho+\lambda) b_1 &= \log(\theta_1) + \frac{1}{\rho}\nu_1 + \lambda b_0.
\end{aligned}$$

We plug in $\rho b_0 = \log(\theta_0) + \frac{1}{\rho}\nu_0$ to get

$$\begin{aligned}
(\rho+\lambda) b_1 &= \log(\theta_1) + \frac{1}{\rho}\nu_1 + \frac{\lambda}{\rho}\left(\log(\theta_0) + \frac{1}{\rho}\nu_0\right), \\
(\rho+\lambda)\rho b_1 &= \rho\log(\theta_1) + \nu_1 + \lambda\left(\log(\theta_0) + \frac{1}{\rho}\nu_0\right), \\
(\rho+\lambda)\rho b_1 &= \rho\log(\theta_1) + \lambda\log(\theta_1) - \lambda\log(\theta_1) + \nu_1 + \lambda\left(\log(\theta_0) + \frac{1}{\rho}\nu_0\right), \\
(\rho+\lambda)\rho b_1 &= (\rho+\lambda)\log(\theta_1) + \nu_1 + \lambda\left(\log(\theta_0) - \log(\theta_1) + \frac{1}{\rho}\nu_0\right), \\
(\rho+\lambda)\rho b_1 &= (\rho+\lambda)\log(\theta_1) + \nu_1 + \frac{\lambda}{\rho}\nu_1 - \frac{\lambda}{\rho}\nu_1 + \lambda\left(\log(\theta_0) - \log(\theta_1) + \frac{1}{\rho}\nu_0\right), \\
(\rho+\lambda)\rho b_1 &= (\rho+\lambda)\log(\theta_1) + \frac{\rho+\lambda}{\rho}\nu_1 + \lambda\left(\log(\theta_0) - \log(\theta_1) + \frac{1}{\rho}(\nu_0 - \nu_1)\right), \\
\rho b_1 &= \log(\theta_1) + \frac{1}{\rho}\nu_1 - \frac{\lambda}{\rho+\lambda}\left(\log(\theta_1) - \log(\theta_0) + \frac{1}{\rho}(\nu_1 - \nu_0)\right).
\end{aligned}$$

□

Using $\lambda(b_1 - b_0) = \log(\theta_1) + \frac{1}{\rho}\nu_1 - \rho b_1$, it is useful to write

$$(b_1 - b_0) = \frac{1}{\rho+\lambda}\left(\log(\theta_1) - \log(\theta_0) + \frac{1}{\rho}\left(\nu_1 - \nu_0\right)\right). \tag{45}$$

**Adding Sustainability Constraint** The analysis with the sustainability constraint in partial equilibrium is no different from in the previous section. What changes is just the welfare weighting of the Markov risk, as already noted. The expected change in welfare is the drift in wealth minus the Itô term reflecting risk plus the Poisson term.

$$\mathbb{E}[\mathrm{d}V_i] = \frac{1}{\rho}\left(r + \alpha_i\mu_i - \theta_i - \frac{1}{2}\alpha_i^2\sigma^2\right) - \lambda(b_i - b_j) = \frac{\nu_i}{\rho} - \lambda(b_i - b_j).$$

**Proposition 14.** *The sustainability constraint will bind whenever*

$$r + \frac{1}{2}SR_i^2 - \frac{\lambda}{2(\rho+\lambda)}\left(SR_i^2 - SR_j^2\right) < \rho. \tag{46}$$

*Proof.* Let's first go directly to the optimized conditions, in which case $\theta_i = \theta_j = \rho$ and $\nu_i - \nu_j = \frac{1}{2}(SR_i^2 - SR_j^2)$. From equation (43)

$$\mathbb{E}[\mathrm{d}V_i] = r - \rho + \frac{1}{2}SR_i^2 - \frac{\lambda}{2(\rho+\lambda)}\left(SR_i^2 - SR_j^2\right).$$

□

When the constraint binds, we then have the following condition on $\theta_i, \alpha_i$:

$$r + \alpha_i\mu_i - \theta_i - \frac{1}{2}\alpha_i^2\sigma^2 = \rho\lambda(b_i - b_j), \implies \nu_i = \rho\lambda(b_i - b_j). \tag{47}$$

**Proposition 15.** *When the SC binds, the optimal portfolio is unchanged, $\alpha_i = \frac{\mu_i}{\sigma^2}$, and consumption solves*

$$\theta_i + \frac{\rho\lambda}{\rho+\lambda}\log(\theta_i) = r + \frac{1}{2}SR_i^2 + \frac{\rho\lambda}{\rho+\lambda}\left(\log(\rho) + (r - \rho + \frac{1}{2}SR_j^2)/\rho\right). \tag{48}$$

*This means that $\theta < \rho$, i.e., we need a lower $\theta$ to satisfy the constraint.*

*Proof.* By solving for $b_i - b_j$ from the sustainability constraint, we can plug into equation (35) to write $b_i$ as

$$\rho b_i = \log(\theta_i) + \frac{1}{\rho}\nu_i - \frac{1}{\rho}(\nu_i) = \log(\theta_i).$$

Maximizing welfare comes down to maximizing $\theta$ subject to the sustainability constraint. The difference is that our sustainability constraint is not so simple, as noted above. Use $\nu_i = \rho\lambda(b_i - b_j)$

from the sustainability constraint with equation (45):

$$\begin{aligned}
\nu_i &= \frac{\rho\lambda}{\rho+2\lambda}\left(\log(\theta_i)-\log(\theta_j)+\frac{1}{\rho}\left(\nu_i-\nu_j\right)\right),\\
\nu_i &= \frac{\rho\lambda}{\rho+2\lambda}\left(\log(\theta_i)-\log(\theta_j)-\nu_j/\rho\right)+\frac{\lambda}{\rho+2\lambda}\nu_i,\\
\nu_i\left(1-\frac{\lambda}{\rho+2\lambda}\right) &= \frac{\rho\lambda}{\rho+2\lambda}\left(\log(\theta_i)-\log(\theta_j)-\nu_j/\rho\right),\\
\nu_i\left(\frac{\rho+2\lambda-\lambda}{\rho+2\lambda}\right) &= \frac{\rho\lambda}{\rho+2\lambda}\left(\log(\theta_i)-\log(\theta_j)-\nu_j/\rho\right),\\
\nu_i\left(\frac{\rho+\lambda}{\rho+2\lambda}\right) &= \frac{\rho\lambda}{\rho+2\lambda}\left(\log(\theta_i)-\log(\theta_j)-\nu_j/\rho\right),\\
\nu_i &= \frac{\rho\lambda}{\rho+\lambda}\left(\log(\theta_i)-\log(\theta_j)-\nu_j/\rho\right).
\end{aligned}$$

This means we have

$$r+\alpha_i\mu_i-\theta_i-\frac{1}{2}\alpha_i^2\sigma^2=\frac{\rho\lambda}{\rho+\lambda}\left(\log(\theta_i)-\log(\theta_j)-\nu_j/\rho\right),$$

or,

$$\theta_i+\frac{\rho\lambda}{\rho+\lambda}\log(\theta_i)=r+\alpha_i\mu_i-\frac{1}{2}\alpha_i^2\sigma^2+\frac{\rho\lambda}{\rho+\lambda}\left(\log(\theta_j)+\nu_j/\rho\right). \tag{49}$$

Recall that we want to maximize $\log\theta$ (i.e., maximize $\theta$) subject to the constraint. The LHS is increasing in $\theta$, which means we want to maximize the RHS. Hence,

$$\alpha_i=\frac{\mu_i}{\sigma^2},$$

the same FOC from earlier. The portfolio decision is not distorted, same as before. Thus, the consumption rate implicitly solves

$$\theta_i+\frac{\rho\lambda}{\rho+\lambda}\log(\theta_i)=r+\frac{1}{2}SR_i^2+\frac{\rho\lambda}{\rho+\lambda}\left(\log(\rho)+(r-\rho+\frac{1}{2}SR_j^2)/\rho\right), \tag{50}$$

where have substituted in state $j$. □

**General Equilibrium**

*Proof of Proposition 5.* We need to characterize the wedge $\rho\lambda(b_i-b_j)$ in GE. From equation (44), we have

$$\rho\lambda(b_1 - b_0) = \frac{\rho\lambda}{\rho+\lambda}\left(\log(\theta_1) - \log(\theta_0) + \frac{1}{\rho}\left(\nu_1 - \nu_0\right)\right).$$

Thus, in GE and plugging in optimal consumption we have

$$\rho\lambda(b_1 - b_0) = \frac{\lambda}{\rho+\lambda}\left(g_1 - g_0\right).$$

Thus, sustainability will bind in 1 if $g_1 - \frac{1}{2}\sigma^2 - \frac{\lambda}{\rho+\lambda}\left(g_1 - g_0\right) < 0$. □